2-Cocycles of Deformative Schrödinger-Virasoro Algebras[1]

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Abstract. In a series of papers by Henkel, Roger and Unterberger, Schrödinger-Virasoro algebras and their deformations were introduced and investigated. In the present paper we determine the 2-cocycles of a class of deformative Schrödinger-Virasoro algebras.

Key words: Schrödinger-Virasoro algebras, 2-cocycles.

§1. Introduction

It is well known that the infinite-dimensional Schrödinger Lie algebras and Virasoro algebra play important roles in many areas of mathematics and physics (e.g., statistical physics). The Schrödinger-Virasoro algebras and their deformations were introduced in [2, 3, 4, 9], in the context of non-equilibrium statistical physics, closely related to both Schrödinger Lie algebras and the Virasoro Lie algebra. Their vertex algebra representations were constructed in [15], and later the derivation algebra and the automorphism group of the twisted sector were determined in [5]. Furthermore, irreducible modules with finite-dimensional weight spaces and indecomposable modules over both original and twisted sectors were investigated in [6].

The infinite-dimensional Lie algebras $\mathcal{L}_{l,\mu}\,(l,\mu\in\mathbb{C})$ considered in this paper called *twisted deformative Schrödinger-Virasoro Lie algebras* (see [9]), possess the same $\mathbb{C}$-basis

$$\{L_n,\, M_n,\, Y_n\,|\,n\in\mathbb{Z}\}$$

with the following Lie brackets:

$$[L_n,L_m]=(m-n)L_{m+n}, \tag{1.1}$$

$$[L_n,Y_m]=(m-\frac{(l+1)n}{2}+\mu)Y_{m+n},\quad [Y_n,Y_m]=(m-n)M_{m+n}, \tag{1.2}$$

$$[L_n,M_m]=(m-ln+2\mu)M_{m+n},\qquad [Y_n,M_m]=[M_n,M_m]=0. \tag{1.3}$$

The purpose of this paper is to determine the 2-cocycles of deformative Schrödinger-Virasoro algebras $\mathcal{L}_{l,\mu}\,(l,\mu\in\mathbb{C})$ defined above. The 2-cocycles on Lie algebras play important roles in the central extensions of Lie algebras, which can be used to construct many

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infinite-dimensional Lie algebras, such as affine Lie algebras, Heisenberg algebras with a profound mathematical and physical background, and further to describe the structures and some of the representations of these Lie algebras. It is well known that all 1-dimensional central extensions of some $\mathcal{L}_{l,\mu}$ determine its 2-cohomology group. Since the cohomology groups are closely related to the structure of Lie algebras, the computation of cohomology groups seems to be important and interesting as well. Partially due to the reasons stated above, there appeared a number of papers on 2-cocycles and cohomology groups of infinite-dimensional Lie algebras and conformal algebras (see [1], [7, 8], [10]–[14] and related references cited in those papers). Now let's formulate our main results below.

We start with a brief definition. Recall that a *2-cocycle* on some $\mathcal{L}_{l,\mu}$ is a $\mathbb{C}$-bilinear function $\psi : \mathcal{L}_{l,\mu} \times \mathcal{L}_{l,\mu} \longrightarrow \mathbb{C}$ satisfying the following conditions:

$$\begin{aligned}
&\psi(v_1, v_2) = -\psi(v_2, v_1) \quad (\text{skew-symmetry}),\\
&\psi([v_1, v_2], v_3) + \psi([v_2, v_3], v_1) + \psi([v_3, v_1], v_2) = 0 \quad (\text{Jacobian identity}),
\end{aligned} \tag{1.4}$$

for $v_1, v_2, v_3 \in \mathcal{L}_{l,\mu}$. Denote by $\mathcal{C}^2(\mathcal{L}_{l,\mu}, \mathbb{C})$ the vector space of 2-cocycles on $\mathcal{L}_{l,\mu}$. For any $\mathbb{C}$-linear function $f : \mathcal{L}_{l,\mu} \longrightarrow \mathbb{C}$, one can define a 2-cocycle ψ_f as follows

$$\psi_f(v_1, v_2) = f([v_1, v_2]), \qquad \forall\, v_1, v_2 \in \mathcal{L}_{l,\mu}. \tag{1.5}$$

Such a 2-cocycle is called a *2-coboundary* or a *trivial 2-cocycle* on $\mathcal{L}_{l,\mu}$. Denote by $\mathcal{B}^2(\mathcal{L}_{l,\mu}, \mathbb{C})$ the vector space of 2-coboundaries on $\mathcal{L}_{l,\mu}$. A 2-cocycle φ is said to be *equivalent to* a 2-cocycle ψ if $\varphi - \psi$ is trivial. For a 2-cocycle ψ, we denote by $[\psi]$ the equivalent class of ψ. The quotient space

$$\mathcal{H}^2(\mathcal{L}_{l,\mu}, \mathbb{C}) = \mathcal{C}^2(\mathcal{L}_{l,\mu}, \mathbb{C})/\mathcal{B}^2(\mathcal{L}_{l,\mu}, \mathbb{C}) = \{\text{the equivalent classes of 2-cocycles}\},$$

is called the *second cohomology group* of $\mathcal{L}_{l,\mu}$.

For the case $\mu = 0$, this problem has been considered and solved in [9] by using the homological method. So we only need to consider the case $\mu \in \mathbb{C}^*$ in the present paper. We call a 2-cocycle ξ on $\mathcal{L}_{l,\mu}$ the *Virasoro cocycle*, denoted by ξ_{Vir}, if

$$\xi(L_n, L_m) = \frac{n^3 - n}{12}\delta_{m,-n}\,, \quad \text{while other components vanishing.} \tag{1.6}$$

The main results of the paper can be formulated as the following theorem.

Theorem 1.1. *(i) If $\mu \notin \{\frac{1}{3}\mathbb{Z}\}$, then for any $l \in \mathbb{C}$, $\mathcal{H}^2(\mathcal{L}_{l,\mu}, \mathbb{C}) \cong \mathbb{C}$ is generated by the Virasoro cocycle.*
(ii) If $\mu \in \{\frac{1}{3} + \mathbb{Z}, \frac{2}{3} + \mathbb{Z}\}$, then for any $l \neq -1$, also $\mathcal{H}^2(\mathcal{L}_{l,\mu}, \mathbb{C}) \cong \mathbb{C}$ is generated by the Virasoro cocycle.
(iii) If $\mu \in \{\frac{1}{3} + \mathbb{Z}, \frac{2}{3} + \mathbb{Z}\}$ and $l = -1$, $\mathcal{H}^2(\mathcal{L}_{l,\mu}, \mathbb{C}) \cong \mathbb{C}^2$ is generated by the Virasoro cocycle

and an independent cocycle of the form $c(M_n, Y_m) = \delta_{n,-m-3\mu}$.
(iv) For $\mu \in \mathbb{Z}^*$ *and* $l \neq -3, -1, 1$, $\mathcal{H}^2(\mathcal{L}_{l,\mu}, \mathbb{C}) \cong \mathbb{C}$ *is generated by the Virasoro cocycle.*
(v) For $\mu \in \mathbb{Z}^*$ *and* $l = -3, 1$, $\mathcal{H}^2(\mathcal{L}_{l,\mu}, \mathbb{C}) \cong \mathbb{C}^2$ *is generated by the Virasoro cocycle and an independent cocycle of the form* $c(L_n, Y_m) = \frac{m+\mu+1}{2}\delta_{n,-m-\mu}$ *for* $l = -3$ *or* $c(L_n, Y_m) = (m+\mu-1)(m+\mu)(m+\mu+1)\delta_{n,-m-\mu}$ *for* $l = 1$.
(vi) For $\mu \in \mathbb{Z}^*$ *and* $l = -1$, $\mathcal{H}^2(\mathcal{L}_{l,0}, \mathbb{C}) \cong \mathbb{C}^3$ *is generated by the Virasoro cocycle and other two independent cocycles* c_1 *and* c_2 *defined by (all other components vanishing)*

$$c_1(L_n, Y_m) = \frac{(m+\mu)(m+\mu+1)}{2}\delta_{n,-m-\mu}; \quad c_2(M_n, Y_m) = \delta_{n,-m-3\mu}.$$

Throughout the paper, we denote by $\mathbb{Z}^*$ the set of all nonzero integers, $\mathbb{C}^*$ the set of all nonzero complex numbers and $\mathbb{C}^*\backslash\mathbb{Z}^* = \{x \,|\, x \in \mathbb{C}^*, \, x \notin \mathbb{Z}^*\}$.

§2. Proof the main results

Let ψ be any 2-cocycle. Our main object is to obtain all equivalent classes of the nontrivial 2-cocycles by means of subtracting all equivalent classes of the 2-coboundaries on $\mathcal{L}_{l,\mu}$ from ψ.

Define a $\mathbb{C}$-linear function $f : \mathcal{L}_{l,\mu} \to \mathbb{C}$ as follows

$$f(L_n) = \begin{cases} \frac{1}{n}\psi(L_0, L_n) & \text{if } \; n \neq 0, \; \forall \; \mu \in \mathbb{C}^*, \\ \frac{1}{2}\psi(L_{-1}, L_1) & \text{if } \; n = 0, \; \forall \; \mu \in \mathbb{C}^*, \end{cases} \tag{2.1}$$

$$f(M_n) = \begin{cases} \frac{1}{n+2\mu}\psi(L_0, M_n) & \text{if } \; n \neq -2\mu, \; \mu \in \frac{1}{2} + \mathbb{Z} \cup \mathbb{Z}^*, \; \text{or } \mu \notin \frac{1}{2}\mathbb{Z}, \\ \frac{-1}{l+1}\psi(L_1, M_{-2\mu-1}) & \text{if } \; n = -2\mu, \; l \neq -1, \; \text{and } \mu \in \frac{1}{2} + \mathbb{Z} \cup \mathbb{Z}^*, \end{cases} \tag{2.2}$$

$$f(Y_n) = \begin{cases} \frac{1}{n+\mu}\psi(L_0, Y_n) & \text{if } \; n \neq -\mu, \; \mu \in \mathbb{Z}^*, \; \text{or } \mu \in \mathbb{C}^*\backslash\mathbb{Z}^*, \\ \frac{-2}{l+3}\psi(L_1, Y_{-\mu-1}) & \text{if } \; n = -\mu, l \neq -3 \text{ and } \mu \in \mathbb{Z}^*. \end{cases} \tag{2.3}$$

Let $\varphi = \psi - \psi_f - \xi_{Vir}$ where ψ_f and ξ_{Vir} are respectively defined in (1.5) and (1.6), then

$$\varphi(L_m, L_n) = 0, \quad \forall \; m, \, n \in \mathbb{Z}. \tag{2.4}$$

Case 1. $\mu \notin \frac{1}{2}\mathbb{Z}$.

Lemma 2.1. *If* $\mu \in \{\frac{1}{3} + \mathbb{Z}, \, \frac{2}{3} + \mathbb{Z}\}$ and $l = -1$, *one has (other components vanishing)*

$$\varphi(M_{-m-3\mu}, Y_m) = \varphi(M_{-3\mu}, Y_0), \quad \forall \; m, n \in \mathbb{Z}; \tag{2.5}$$

otherwise, $\varphi = 0$.

Proof. According to (2.2) and (2.3), one has

$$\varphi(L_0, Y_n) = \varphi(L_0, M_n) = 0 \ \ \forall\, n \in \mathbb{Z}. \tag{2.6}$$

For any $m, n \in \mathbb{Z}$, using the Jacobian identity on the triple $(L_0,\, Y_m,\, Y_n)$, together with (2.4), we obtain

$$(m + n + 2\mu)\varphi(Y_m, Y_n) = 0,$$

which together with our assumption $\mu \notin \frac{1}{2}\mathbb{Z}$ forces

$$\varphi(Y_m, Y_n) = 0. \tag{2.7}$$

Using the Jacobian identity on the four triples $(L_m,\, Y_n,\, L_0)$, $(L_m,\, M_n,\, L_0)$, $(Y_m,\, M_n,\, L_0)$ and $(M_m,\, M_n,\, L_0)$ in (1.4) respectively, one has

$$(\, m + n + \mu\,)\, \varphi(\, L_m, Y_n\,) \ = 0, \tag{2.8}$$

$$(m + n + 2\mu)\, \varphi(L_m, M_n) = 0, \tag{2.9}$$

$$(m + n + 3\mu)\, \varphi(Y_m, M_n) \ = 0, \tag{2.10}$$

$$(m + n + 4\mu)\varphi(M_m, M_n) = 0. \tag{2.11}$$

Then our assumption $\mu \notin \frac{1}{2}\mathbb{Z}$, together with (2.8) and (2.9) gives

$$\varphi(L_m, Y_n) = \varphi(L_m, M_n) = 0. \tag{2.12}$$

Subcase 1.1. $\mu \notin \{\frac{1}{3} + \mathbb{Z},\, \frac{2}{3} + \mathbb{Z},\, \frac{1}{4} + \mathbb{Z},\, \frac{3}{4} + \mathbb{Z}\}$.

In this subcase, (2.10) and (2.11) force

$$\varphi(Y_m, M_n) = \varphi(M_m, M_n) = 0. \tag{2.13}$$

Subcase 1.2. $\mu \in \{\frac{1}{3} + \mathbb{Z}\} \bigcup \{\frac{2}{3} + \mathbb{Z}\}$.

In this subcase, $m + n + 4\mu \neq 0$ for any $m, n \in \mathbb{Z}$. Then (2.10) and (2.11) imply

$$\varphi(\, Y_m, M_n\,) = 0 \quad \text{ if } \ m + n \neq -3\mu, \tag{2.14}$$

$$\varphi(M_m, M_n) = 0 \quad \text{ for any } \ m,\, n \in \mathbb{Z}. \tag{2.15}$$

We only need to compute the value of

$$\varphi(M_{-3\mu-m}, Y_m) \quad \text{for any } m \in \mathbb{Z}.$$

Using the Jacobian identity on the triple $(L_{-m},\, Y_m,\, M_{-3\mu})$, one has

$$\big((l + 3)m + 2\mu\big)\varphi(M_{-3\mu}, Y_0) + 2(lm - \mu)\varphi(M_{-m-3\mu}, Y_m) = 0. \tag{2.16}$$

If $l=0$, then (2.16) gives (since $\mu\neq 0$ in this case)

$$\varphi(M_{-m-3\mu},Y_m)=(\frac{3m}{2\mu}+1)\varphi(M_{-3\mu},Y_0). \tag{2.17}$$

Applying the Jacobian identity on the triple $(L_{-m-3\mu},Y_m,M_0)$ for any $m\in\mathbb{Z}$, together with (2.17), one has $(m+3\mu)\varphi(M_{-3\mu},Y_0)=0$, which gives $\varphi(M_{-3\mu},Y_0)=0$, and further $\big($together with (2.17)$\big)$

$$\varphi(M_{-m-3\mu},Y_m)=0. \tag{2.18}$$

If $l\in\mathbb{C}^*$ and $\frac{\mu}{l}\notin\mathbb{Z}$, then (2.16) gives

$$\varphi(M_{-m-3\mu},Y_m)=\frac{(l+3)m+2\mu}{2(\mu-ml)}\varphi(M_{-3\mu},Y_0). \tag{2.19}$$

Applying the Jacobian identity on the triple $(L_{-m-3\mu},Y_m,M_0)\,(\forall\ m\in\mathbb{Z})$, together with (2.19), one has (our assumption forcing $l\neq-\frac{1}{3}$)

$$\frac{9(1+l)(m+3\mu)\big(ml(3+l)+\mu(l-1)\big)}{4(1+3l)(ml-mu)}\varphi(M_{-3\mu},Y_0)=0,$$

which gives $\varphi(M_{-3\mu},Y_0)=0$ if $l\neq-1$ and further $\big($together with (2.19)$\big)$

$$\varphi(M_{-m-3\mu},Y_m)=0\quad \text{if } l\neq-1. \tag{2.20}$$

For the case $l=-1$, (2.19) gives

$$\varphi(M_{-m-3\mu},Y_m)=\varphi(M_{-3\mu},Y_0). \tag{2.21}$$

If $l\in\mathbb{C}^*$ while $\frac{\mu}{l}\in\mathbb{Z}$, then (2.16) gives (since$l\neq-1,-3$ in this subcase)

$$\begin{aligned}&(1+\frac{1}{l})\mu\varphi(M_{-3\mu},Y_0)=0\ \text{ and }\ \varphi(M_{-3\mu},Y_0)=0,\\ &\varphi(M_{-m-3\mu},Y_m)=\frac{(l+3)m+2\mu}{2(\mu-ml)}\varphi(M_{-3\mu},Y_0)=0\quad\text{if}\ \ m\neq\frac{\mu}{l}.\end{aligned} \tag{2.22}$$

Using the Jacobian identity on $(L_1,\,Y_{\frac{\mu}{l}-1},\,M_{-\frac{\mu}{l}-3\mu})$ and $(L_2,\,Y_{\frac{\mu}{l}-2},\,M_{-\frac{\mu}{l}-3\mu})$, together with (2.22), one has

$$(l+3)\varphi(M_{-\frac{\mu}{l}-3\mu},Y_{\frac{\mu}{l}})=0\ \text{ i.e., }\ \varphi(M_{-\frac{\mu}{l}-3\mu},Y_{\frac{\mu}{l}})=0\ (\text{as }\ l\neq-3),$$

which combined with (2.22) gives

$$\varphi(M_{-3\mu-m},Y_m)=0,\quad \forall\ m\in\mathbb{Z}. \tag{2.23}$$

Subcase 1.3. $\mu\in\{\frac{1}{4}+\mathbb{Z}\}\bigcup\{\frac{3}{4}+\mathbb{Z}\}$.

In this subcase, $m+n+3\mu\neq 0$ for any $m,n\in\mathbb{Z}$. Then (2.10) and (2.11) give

$$\varphi(M_m,M_n)=0 \quad \text{if } \ m+n\neq -4\mu, \tag{2.24}$$
$$\varphi(\,Y_m,M_n\,)=0 \quad \text{for any } \ m,\,n\in\mathbb{Z}. \tag{2.25}$$

Then the left sector we have to consider in this subcase is $\varphi(M_{-4\mu-m},M_m)$ for any $m\in\mathbb{Z}$.

Using the Jacobian identity on the triple $(Y_{-4\mu},\,Y_{-m},\,M_m)$, one has

$$-(m-4\mu)\varphi(M_{-m-4\mu},M_m)=0,$$

which gives

$$\varphi(M_{-m-4\mu},M_m)=0 \quad \text{for any } \ 4\mu\neq m\in\mathbb{Z}. \tag{2.26}$$

Using the Jacobian identity on the triples $(Y_{-6\mu},\,Y_{-2\mu},\,M_{4\mu})$, one has

$$4\mu\varphi(M_{-8\mu},M_{4\mu})=0,$$

which gives $\varphi(M_{-8\mu},M_{4\mu})=0$ and further together with (2.24) gives

$$\varphi(M_m,M_n)=0,\ \ \forall\ m,n\in\mathbb{Z}. \tag{2.27}$$

Then this lemma follows. $\square$

This lemma in particular proves Theorem 1.1 (i)–(iii) in this case.

Case 2. $\mu\in\frac{1}{2}+\mathbb{Z}$.

Lemma 2.2. *In this case* $\varphi(x,y)=0,\ \ \forall\ l\in\mathbb{C},\ x,y\in\mathcal{L}_{l,\mu}$.

Proof. According to (2.2) and (2.3), one has

$$\varphi(L_0,Y_n)=0\ \ \forall\ n\in\mathbb{Z}, \tag{2.28}$$
$$\varphi(L_0,M_n)=0\ \text{ if }\ n\neq -2\mu,\quad \varphi(L_1,M_{-2\mu-1})=0\ \text{ if }\ l\neq -1. \tag{2.29}$$

Using the Jacobian identity by replacing the triple (v_1,v_2,v_3) by five triples $(L_m,\,Y_n,\,L_0)$, $(Y_m,\,M_n,\,L_0)$, $(M_m,\,M_n,\,L_0)$, $(Y_m,\,Y_n,\,L_0)$ and $(L_m,\,M_n,\,L_0)$ in (1.4) respectively, together with (2.28), one has

$$(m+n+\mu)\varphi(L_m,Y_n)=0, \tag{2.30}$$
$$(m+n+3\mu)\varphi(Y_m,M_n)=0, \tag{2.31}$$
$$(m+n+4\mu)\varphi(M_m,M_n)=0, \tag{2.32}$$
$$(m-n)\varphi(L_0,M_{m+n})+(m+n+2\mu)\varphi(Y_m,Y_n)=0, \tag{2.33}$$
$$(-n+ml-2\mu)\varphi(L_0,M_{m+n})+(m+n+2\mu)\varphi(L_m,M_n)=0. \tag{2.34}$$

In this case, $(m+n+\mu)(m+n+3\mu)\neq 0$. Hence (2.30) and (2.31) force

$$\varphi(L_m,Y_n)=\varphi(Y_m,M_n)=0. \tag{2.35}$$

According to (2.29), (2.33) and (2.34), one can deduce

$$\varphi(Y_m,Y_n)=0 \quad \text{if } m+n\neq -2\mu, \tag{2.36}$$

$$\varphi(L_m,M_n)=0 \quad \text{if } m+n\neq -2\mu, \tag{2.37}$$

$$-2(n+\mu)\varphi(L_0,M_{-2\mu})=0 \quad \text{if } m+n=-2\mu. \tag{2.38}$$

The assumption $\mu\in\frac{1}{2}+\mathbb{Z}$ and (2.38) infer

$$\varphi(L_0,M_{-2\mu})=0. \tag{2.39}$$

From (2.29), (2.28), (2.32), (2.35), (2.36) and (2.39), the left components we have to present in this case are listed in the following (where m is an arbitrary integer):

$$\varphi(Y_{-2\mu-m},Y_m),\quad \varphi(L_{-2\mu-m},M_m)\ \text{ and }\ \varphi(M_{-4\mu-m},M_m).$$

Repeating the proving process between (2.24) and (2.27), in this case one also can obtain

$$\varphi(M_{-4\mu-m},M_m)=0,\quad \forall\ m\in\mathbb{Z}. \tag{2.40}$$

Using the Jacobian identity on $(L_{-m},\,Y_{-2\mu},\,Y_m)$ and $(L_{-2\mu-m},\,L_n,\,M_{m-n})$ ($\forall\ n\in\mathbb{Z}$) respectively, one has

$$\begin{aligned}&\big(m(1+l)-2\mu\big)\varphi(Y_{-m-2\mu},Y_m)\\&=2(m+2\mu)\varphi(L_{-m},M_{m-2\mu})-\big(m(3+l)+2\mu\big)\varphi(Y_{-2\mu},Y_0),\end{aligned} \tag{2.41}$$

$$\begin{aligned}&\big(m+2\mu-n(1+l)\big)\varphi(L_{-m-2\mu},M_m)\\&=\big((m+2\mu)(1+l)-n\big)\varphi(L_n,M_{-n-2\mu})+(m+2\mu+n)\varphi(L_{n-m-2\mu},M_{m-n}).\end{aligned} \tag{2.42}$$

Replacing n by $-n$ and m by $m+n$ in (2.42), one respectively gets

$$\begin{aligned}&\big(m+2\mu+n(1+l)\big)\varphi(L_{-m-2\mu},M_m)\\&=(m+2\mu-n)\varphi(L_{-n-m-2\mu},M_{m+n})+\big((m+2\mu)(1+l)+n\big)\varphi(L_{-n},M_{n-2\mu}),\end{aligned} \tag{2.43}$$

$$\begin{aligned}&(2n+m+2\mu)\varphi(L_{-m-2\mu},M_m)\\&=(m+2\mu-nl)\varphi(L_{-n-m-2\mu},M_{m+n})-\big((n+m+2\mu)(1+l)-n\big)\varphi(L_n,M_{-n-2\mu}).\end{aligned} \tag{2.44}$$

According to (2.29), we have to divide the left part of this proof into two subcases.

Subcase 2.1. $l\neq -1$.

In this subcase, taking $n=-1$ in both (2.43) and (2.44), together with (2.29), we have

$$(m+2\mu-1-l)\varphi(L_{-m-2\mu},M_m)=(m+2\mu+1)\varphi(L_{1-m-2\mu},M_{m-1}), \tag{2.45}$$

$$\begin{aligned}&(m+2\mu-2)\varphi(L_{-m-2\mu},M_m)\\&=(m+2\mu+l)\varphi(L_{1-m-2\mu},M_{m-1})-\big((m+2\mu-1)(1+l)+1\big)\varphi(L_{-1},M_{1-2\mu}).\end{aligned} \tag{2.46}$$

If $(l^2+l-2)(1+l)\neq 0$, then combining (2.45) with (2.46), one can deduce

$$\varphi(L_{-m-2\mu},M_m)=-\frac{(m+2\mu+1)\big((m+2\mu)(1+l)-l\big)}{l^2+l-2}\varphi(L_{-1},M_{1-2\mu}). \tag{2.47}$$

Taking $\varphi(L_{-m-2\mu},M_m)$ and $\varphi(L_{1-m-2\mu},M_{m-1})$ obtained from (2.47) back to (2.45), one has

$$\frac{l(1+l)\big((m+2\mu)^2-1\big)\varphi(L_{-1},M_{1-2\mu})}{l^2+l-2}=0,$$

which forces (since the index m can be shifted and our assumption $l\neq -1$)

$$l\varphi(L_{-1},M_{1-2\mu})=0.$$

In another word,

> the system consisted of linear equations (2.45) and (2.46) has nonzero solutions if and only if $l=0$ under our assumption $(l^2+l-2)(1+l)\neq 0$. (2.48)

Hence based on the discussions between (2.45) and (2.48), we have to divide Subcase 2.1 into another four subcases.

Subcase 2.1(i). $l=0$.

If $l=0$, then (2.47) can be rewritten as

$$\varphi(L_{-m-2\mu},M_m)=\frac{(m+2\mu)(m+2\mu+1)}{2}\varphi(L_{-1},M_{1-2\mu}). \tag{2.49}$$

If $l=0$ and $m\neq 2\mu$, then (2.49) together with (2.41) gives

$$\varphi(Y_{-m-2\mu},Y_m)=\frac{m(m+1)(m+2\mu)}{m-2\mu}\varphi(L_{-1},M_{1-2\mu})-\frac{3m+2\mu}{m-2\mu}\varphi(Y_{-2\mu},Y_0). \tag{2.50}$$

Using the Jacobian identity on the triple $(L_{-2\mu+m},\,Y_0,\,Y_{-m})$, one has

$$\begin{aligned}&\big(2u(2+l)-m(1+l)\big)\varphi(Y_{m-2\mu},Y_{-m})\\&=\big(m(3+l)-2\mu(2+l)\big)\varphi(Y_0,Y_{-2\mu})-2m\varphi(L_{m-2\mu},M_{-m}).\end{aligned} \tag{2.51}$$

Taking $l=0$, $m=-2\mu$ in (2.51) and using (2.49) together with (2.50), one has

$$6\mu\varphi(Y_{-4\mu},Y_{2\mu})=10\mu\varphi(Y_{-2\mu},Y_0)+8\mu^2(4\mu+1)\varphi(L_{-1},M_{1-2\mu}), \tag{2.52}$$

which gives

$$\varphi(Y_{-4\mu},Y_{2\mu})=\frac{5}{3}\varphi(Y_{-2\mu},Y_0)+\frac{4\mu(4\mu+1)}{3}\varphi(L_{-1},M_{1-2\mu}). \tag{2.53}$$

For any $p \in \mathbb{Z}^*$, applying the Jacobian identity on the triple $(L_p, Y_{2\mu}, Y_{-p-4\mu})$, together with (2.49), (2.50) and (2.53), we obtain

$$(p+4\mu)\Big((p+6\mu)\varphi(Y_{-2\mu}, Y_0) + \big(p^3 - 6(1+2\mu)\mu^2 - p\mu(1+6\mu)\big)\varphi(L_{-1}, M_{1-2\mu})\Big) = 0,$$

which forces

$$\varphi(Y_{-2\mu}, Y_0) = \varphi(L_{-1}, M_{1-2\mu}) = 0.$$

and further $\big($recalling (2.50) and (2.53)$\big)$

$$\varphi(Y_{-m-2\mu}, Y_m) = 0, \quad \forall\, m \in \mathbb{Z}. \tag{2.54}$$

Subcase 2.1(ii). $l = 1$.

If $l = 1$, then (2.45) becomes

$$(m+2\mu-2)\varphi(L_{-m-2\mu}, M_m) = (m+2\mu+1)\varphi(L_{1-m-2\mu}, M_{m-1}), \tag{2.55}$$

which further gives $\big($by taking $m = 2 - 2\mu$ in (2.55)$\big)$

$$\varphi(L_{-1}, M_{1-2\mu}) = 0. \tag{2.56}$$

Also by (2.29), (2.39), (2.55) and (2.56), one can deduce

$$\varphi(L_{-m-2\mu}, M_m) = \begin{cases} (m+2\mu-1)(m+2\mu)(m+2\mu+1)c_1 & \text{if } m \geq -2\mu-1, \\ (m+2\mu-1)(m+2\mu)(m+2\mu+1)c_2 & \text{if } m < -2\mu-1, \end{cases} \tag{2.57}$$

for some constants $c_1,\ c_2 \in \mathbb{C}$. One thing left to be done is to find the relations between the constants c_1 and c_2. If $l = 1$, then (2.43) and (2.44) become

$$\begin{aligned}
&(m+2\mu+2n)\varphi(L_{-m-2\mu}, M_m) \\
&= (m+2\mu-n)\varphi(L_{-n-m-2\mu}, M_{m+n}) + \big(2(m+2\mu)+n\big)\varphi(L_{-n}, M_{n-2\mu}), \\
&(m+2\mu+2n)\varphi(L_{-m-2\mu}, M_m) \\
&= (m+2\mu-n)\varphi(L_{-n-m-2\mu}, M_{m+n}) - \big(2(m+2\mu)+n\big)\varphi(L_n, M_{-n-2\mu}),
\end{aligned}$$

which together with each other force

$$\big(2(m+2\mu)+n\big)\big(\varphi(L_{-n}, M_{n-2\mu}) + \varphi(L_n, M_{-n-2\mu})\big) = 0, \tag{2.58}$$

and in particular give (by taking $n = 2$)

$$\big(m+2\mu+1\big)\big(\varphi(L_{-2}, M_{2-2\mu}) + \varphi(L_2, M_{-2-2\mu})\big) = 0. \tag{2.59}$$

Noticing $2-2\mu > -1-2\mu$, $-2-2\mu < -1-2\mu$, and then combining (2.57) with (2.59), one can safely deduce

$$c_1 = c_2,$$

which together with (2.57) gives

$$\varphi(L_{-m-2\mu}, M_m) = (m+2\mu-1)(m+2\mu)(m+2\mu+1)c_1, \quad \forall\, m \in \mathbb{Z}. \tag{2.60}$$

If $l = 1$, then $m \neq \mu$ ($\forall\, m \in \mathbb{Z}$) and (2.60) together with (2.41) gives

$$\varphi(Y_{-m-2\mu}, Y_m) = \frac{\mu+2m}{\mu-m}\varphi(Y_{-2\mu}, Y_0). \tag{2.61}$$

For any $p \in \mathbb{Z}^*$, applying the Jacobian identity on the triple $(L_p, Y_{2\mu}, Y_{-p-4\mu})$, together with (2.60) and (2.61), one has

$$p(p+4\mu)\varphi(Y_{-2\mu}, Y_0) - 5c_1(p^2-1)(p^2+11p\mu+30\mu^2) = 0,$$

which implies $\varphi(Y_{-2\mu}, Y_0) = c_1 = 0$ and further

$$\varphi(L_{-m-2\mu}, M_m) = \varphi(Y_{-m-2\mu}, Y_m) = 0. \tag{2.62}$$

Subcase 2.1(iii). $l = -2$.

If $l = -2$, then (2.43) and (2.44) convert to the following form:

$$\begin{aligned}
&(m+2\mu-n)\big(\varphi(L_{-m-2\mu}, M_m) - \varphi(L_{-n-m-2\mu}, M_{m+n})\big)\\
&= (n-m-2\mu)\varphi(L_{-n}, M_{n-2\mu}), \qquad (2.63)\\
&(m+2\mu+2n)\big(\varphi(L_{-m-2\mu}, M_m) - \varphi(L_{n-m-2\mu}, M_{m+n})\big)\\
&= (m+2\mu+2n)\varphi(L_n, M_{-n-2\mu}). \qquad (2.64)
\end{aligned}$$

Furthermore, taking $n = -1$ in both (2.63) and (2.64), and using (2.29), one has

$$\begin{aligned}
&(m+2\mu+1)\big(\varphi(L_{-m-2\mu}, M_m) - \varphi(L_{1-m-2\mu}, M_{m-1})\big) = 0,\\
&(m+2\mu-2)\big(\varphi(L_{-m-2\mu}, M_m) - \varphi(L_{1-m-2\mu}, M_{m-1})\big) = (m+2\mu-1)\varphi(L_{-1}, M_{1-2\mu}),
\end{aligned}$$

from which and using (2.29) again, can we deduce the following relation:

$$\varphi(L_{-m-2\mu}, M_m) = 0, \quad \forall\, m \in \mathbb{Z}. \tag{2.65}$$

If $l = -2$ and $m \neq -2\mu$, then (2.65) together with (2.41) gives

$$\varphi(Y_{-m-2\mu}, Y_m) = \varphi(Y_{-2\mu}, Y_0). \tag{2.66}$$

Using the Jacobian identity on the triple $(L_{-2\mu}, Y_m, Y_{-m})$, one has

$$\begin{aligned}&\big(\mu(2+l)+m\big)\varphi(Y_{m-2\mu}, Y_{-m})\\&=-2m\varphi(L_{-2\mu}, M_0)-\big(\mu(2+l)-m\big)\varphi(Y_m, Y_{-m-2\mu}).\end{aligned}\tag{2.67}$$

Taking $l=-2$, $m=2\mu$ in (2.67) and using (2.65) together with (2.66), one has

$$\varphi(Y_0, Y_{-2\mu})=\varphi(Y_{2\mu}, Y_{-4\mu})=\varphi(Y_{-2\mu}, Y_0),\tag{2.68}$$

which gives

$$\varphi(Y_{-m-2\mu}, Y_m)=\varphi(Y_{-2\mu}, Y_0)=0,\quad \forall\ m\in\mathbb{Z}.\tag{2.69}$$

Subcase 2.1(iv). $l\notin\{-2,\ -1,\ 0,\ 1\}$.

If $l\notin\{-2,\ -1,\ 0,\ 1\}$, then $\varphi(L_{-m}, M_{m-2\mu})=0$ ($\forall\ m\in\mathbb{Z}$) and (2.41) can be rewritten as

$$\big(m(l+1)-2\mu\big)\varphi(Y_{-m-2\mu}, Y_m)=-\big(m(l+3)+2\mu\big)\varphi(Y_{-2\mu}, Y_0).\tag{2.70}$$

If $\frac{2\mu}{l+1}\notin\mathbb{Z}$, then

$$\varphi(Y_{-m-2\mu}, Y_m)=-\frac{m(l+3)+2\mu}{m(l+1)-2\mu}\varphi(Y_{-2\mu}, Y_0).\tag{2.71}$$

For $m=\frac{2\mu}{l+1}\in\mathbb{Z}$, (2.70) gives (since $l\notin\{-2,\ -1,\ 0,\ 1\}$ in this case)

$$\varphi(Y_{-2\mu}, Y_0)=0.\tag{2.72}$$

For $m\neq\frac{2\mu}{l+1}\in\mathbb{Z}$, then

$$\varphi(Y_{-m-2\mu}, Y_m)=\frac{m(l+3)+2\mu}{\mu\big(m(l+1)-2\mu\big)}\varphi(Y_{-2\mu}, Y_0)=0.\tag{2.73}$$

For $l\notin\{-2,\ -1,\ 0,\ 1\}$, taking $m=-\frac{2\mu}{l+1}$ in (2.67) and using (2.48) together with (2.73), one has

$$\begin{aligned}&\frac{\mu(l+1)(l+2)-2\mu}{l+1}\varphi(Y_{-\frac{2\mu}{l+1}-2\mu}, Y_{\frac{2\mu}{l+1}})\\&=\frac{4\mu}{l+1}\varphi(L_{-2\mu}, M_0)-\frac{\mu(l+1)(l+2)+2\mu}{l+1}\varphi(Y_{-\frac{2\mu}{l+1}}, Y_{\frac{2\mu}{l+1}-2\mu})\\&=0,\end{aligned}$$

which gives (since $l\neq-3$ under our assumption $\frac{2\mu}{l+1}\in\mathbb{Z}$)

$$\varphi(Y_{-\frac{2\mu}{l+1}-2\mu}, Y_{\frac{2\mu}{l+1}})=0.\tag{2.74}$$

Subcase 2.2. $l=-1$.

If $l=-1$, $n=1$, then (2.42) can be rewritten as

$$(m+2\mu)\varphi(L_{-m-2\mu},M_m)=(1+m+2\mu)\varphi(L_{1-m-2\mu},M_{m-1})-\varphi(L_1,M_{-1-2\mu}),$$

which is equivalent to

$$\varphi(L_{-m-2\mu},M_m)=\begin{cases}(m+2\mu)\psi(L_1,M_{-1-2\mu}) & \text{if } m\geq -2\mu,\\ (m+2\mu+2)\varphi(L_1,M_{-1-2\mu}) & \\ -(m+2\mu+1)\varphi(L_2,M_{-2-2\mu}) & \text{if } m<-2\mu.\end{cases} \tag{2.75}$$

If $l=-1$, then (2.41) becomes

$$\varphi(Y_{-m-2\mu},Y_m)=\big(\frac{m}{\mu}+1\big)\varphi(Y_{-2\mu},Y_0)-(\frac{m}{\mu}+2)\varphi(L_{-m},M_{m-2\mu}),$$

which combined with (2.75), can be simplified as

$$\begin{aligned}&\varphi(L_{-m-2\mu},Y_m)\\ &=\begin{cases}\big(\frac{m}{\mu}+1\big)\varphi(Y_{-2\mu},Y_0) & \\ -m(\frac{m}{\mu}+2)\psi(L_1,M_{-1-2\mu}) & \text{if } m\geq 0,\\ \big(\frac{m}{\mu}+1\big)\varphi(Y_{-2\mu},Y_0)-(\frac{m}{\mu}+2)\big((m+2) & \\ \times\varphi(L_1,M_{-1-2\mu})-(m+1)\varphi(L_2,M_{-2-2\mu})\big) & \text{if } m<0.\end{cases}\end{aligned} \tag{2.76}$$

However, (2.75) and (2.76) are not compatible with the Jacobian identity given in (1.4), which forces both of them must be zero. This completes the proof of the lemma. □

Then the lemma proves Theorem 1.1 (i) in this case.

Case 3. $\mu\in\mathbb{Z}^*$.

Lemma 2.3. *(i) For the subcase $l\neq -3,-1,1$, one has $\varphi=0$.*
(ii) For the subcase $l=-3$, only $\varphi(L_n,Y_{-n-\mu})$ ($\forall\ n\in\mathbb{Z}$) is not vanishing, given in (2.93).
(iii) For the subcase $l=1$, only $\varphi(L_n,Y_{-n-\mu})$ ($\forall\ n\in\mathbb{Z}$) is not vanishing, given in (2.101).
(iv) For the subcase $l=-1$, only $\varphi(L_n,Y_{-n-\mu})$ and $\varphi(Y_n,M_{-n-3\mu})$ ($\forall\ n\in\mathbb{Z}$)) are not vanishing, which are given in (2.92) and (2.118) respectively.

Proof. One has

$$\varphi(L_0,Y_n)=0\ \text{ if }\ n\neq-\mu,\quad \varphi(L_1,Y_{-\mu-1})=0\ \text{ if }\ l\neq-3, \tag{2.77}$$

$$\varphi(L_0,M_n)=0\ \text{ if }\ n\neq-2\mu,\quad \varphi(L_1,M_{-2\mu-1})=0\ \text{ if }\ l\neq-1. \tag{2.78}$$

Using the Jacobian identity by replacing the triple (v_1, v_2, v_3) by five triples $(Y_m,\ M_n,\ L_0)$, $(M_m,\ M_n,\ L_0)$, $(Y_m,\ Y_n,\ L_0)$, $(L_m,\ Y_n,\ L_0)$ and $(L_m,\ M_n,\ L_0)$ in (1.4) respectively, one has

$$(m+n+3\mu)\varphi(Y_m, M_n) = 0, \tag{2.79}$$
$$(m+n+4\mu)\varphi(M_m, M_n) = 0, \tag{2.80}$$
$$(m-n)\varphi(L_0, M_{m+n}) + (m+n+2\mu)\varphi(Y_m, Y_n) = 0, \tag{2.81}$$
$$(-n+\frac{l+1}{2}m-\mu)\varphi(L_0, Y_{m+n}) + (m+n+\mu)\varphi(L_m, Y_n) = 0, \tag{2.82}$$
$$(-n+ml-2\mu)\varphi(L_0, M_{m+n}) + (m+n+2\mu)\varphi(L_m, M_n) = 0. \tag{2.83}$$

The following results can be directly obtained from (2.81)–(2.83):

$$\varphi(L_0, M_{-2\mu}) = 0 \quad \text{and} \quad (m+n+2\mu)\varphi(Y_m, Y_n) = 0, \tag{2.84}$$
$$\varphi(L_0, M_{-2\mu}) = 0 \quad \text{and} \quad (m+n+2\mu)\varphi(L_m, M_n) = 0, \tag{2.85}$$
$$\varphi(L_0, Y_{-\mu}) = 0 \quad \text{if} \quad l \neq -3, \quad \text{and} \quad (m+n+\mu)\varphi(L_m, Y_n) = 0. \tag{2.86}$$

From (2.77)–(2.78) and (2.84)–(2.86), the left components we have to present in this case are listed in the following (where m is an arbitrary integer):

$$\varphi(L_{-\mu-m}, Y_m),\ \varphi(Y_{-2\mu-m}, Y_m),\ \varphi(L_{-2\mu-m}, M_m),\ \varphi(Y_{-3\mu-m}, M_m) \text{ and } \varphi(M_{-2\mu-m}, M_m),$$

which will be taken into account in the following step by step.

Step 1. *The computation of* $\varphi(L_{-\mu-m}, Y_m)$ $(\forall\ m \in \mathbb{Z})$.

Replacing the triple (v_1, v_2, v_3) by $(L_{-\mu-m},\ L_n,\ Y_{m-n})$ $(\forall\ n \in \mathbb{Z})$ in (1.4), one has

$$\begin{aligned}&\big(2(m+\mu) - n(l+3)\big)\varphi(L_{-m-\mu}, Y_m)\\ &= 2(m+\mu+n)\varphi(L_{n-m-\mu}, Y_{m-n}) + \big((m+\mu)(l+3) - 2n\big)\varphi(L_n, Y_{-n-\mu}),\end{aligned}$$

which gives (by replacing n by $-n$ and m by $m+n$ respectively)

$$\begin{aligned}&\big(2(m+\mu) + n(l+3)\big)\varphi(L_{-m-\mu}, Y_m)\\ &= 2(m+\mu-n)\varphi(L_{-m-\mu-n}, Y_{m+n}) + \big((m+\mu)(l+3) + 2n\big)\varphi(L_{-n}, Y_{n-\mu}),\end{aligned} \tag{2.87}$$
$$\begin{aligned}&\big(2(m+\mu+n) - n(l+3)\big)\varphi(L_{-m-\mu-n}, Y_{m+n})\\ &= 2(m+\mu+2n)\varphi(L_{-m-\mu}, Y_m) + \big((m+\mu+n)(l+3) - 2n\big)\varphi(L_n, Y_{-n-\mu}).\end{aligned} \tag{2.88}$$

If $l \neq -3$, then taking $n = -1$ in both (2.87) and (2.88), together with (2.77), one has

$$\big(2(m+\mu) - (l+3)\big)\varphi(L_{-m-\mu}, Y_m) = 2(m+\mu+1)\varphi(L_{1-m-\mu}, Y_{m-1}), \tag{2.89}$$
$$\begin{aligned}&\big(2(m+\mu-1) + (l+3)\big)\varphi(L_{1-m-\mu}, Y_{m-1})\\ &= 2(m+\mu-2)\varphi(L_{-m-\mu}, Y_m) + \big((m+\mu-1)(l+3) + 2\big)\varphi(L_{-1}, Y_{1-\mu}).\end{aligned} \tag{2.90}$$

If $5-4l-l^2\neq 0$, i. e., $l\neq -5,1$, then combining (2.89) with (2.90), we can deduce

$$\varphi(L_{-m-\mu},Y_m)=\frac{2(m+\mu+1)\big(2+(m+\mu-1)(3+l)\big)\varphi(L_{-1},Y_{1-\mu})}{5-4l-l^2}. \tag{2.91}$$

Taking $\varphi(L_{-m-\mu},Y_m)$ and $\varphi(L_{1-m-\mu},Y_{m-1})$ obtained from (2.91) back to (2.89), one has

$$\frac{2\big((m+\mu)^2-1\big)(l+1)(l+3)\varphi(L_{-1},Y_{1-\mu})}{5-4l-l^2}=0,$$

which forces (since the index m can be shifted and our assumption $l\neq -1$)

$$(l+1)(l+3)\varphi(L_{-1},Y_{1-\mu})=0.$$

In another word, the system consisted of linear equations (2.89) and (2.90) has nonzero solutions if and only if $l=-1$ or $l=-3$ under our assumption $5-4l-l^2\neq 0$.

If $l=-1$, we can write (2.91) as

$$\varphi(L_{-m-\mu},Y_m)=\frac{(m+\mu)(m+\mu+1)}{2}\varphi(L_{-1},Y_{1-\mu}). \tag{2.92}$$

If $l=-3$, we can write (2.91) as

$$\varphi(L_{-m-\mu},Y_m)=\frac{m+\mu+1}{2}\varphi(L_{-1},Y_{1-\mu}). \tag{2.93}$$

If $l=1$, then (2.89) becomes

$$(m+\mu-2)\varphi(L_{-m-\mu},Y_m)=(m+\mu+1)\varphi(L_{1-m-\mu},Y_{m-1}), \tag{2.94}$$

$$\begin{aligned}&(m+\mu+1)\varphi(L_{1-m-\mu},Y_{m-1})\\&=(m+\mu-2)\varphi(L_{-m-\mu},Y_m)+(2m+2\mu-1)\varphi(L_{-1},Y_{1-\mu}),\end{aligned} \tag{2.95}$$

which further gives $\big($by taking $m=-\mu$ in (2.95)$\big)$

$$\varphi(L_{-1},Y_{1-\mu})=0. \tag{2.96}$$

Also by (2.77), (2.86), (2.94) and (2.96), one can deduce

$$\varphi(L_{-m-\mu},Y_m)=\begin{cases}(m+\mu-1)(m+\mu)(m+\mu+1)c_3 & \text{if } m\geq -\mu-1,\\(m+\mu-1)(m+\mu)(m+\mu+1)c_4 & \text{if } m<-\mu-1.\end{cases} \tag{2.97}$$

where c_3, c_4 are some constants in $\mathbb{C}$. One thing left to be done is to find the relations between the constants c_3 and c_4. If $l=1$, then (2.87) and (2.88) become

$$\begin{aligned}&(m+\mu+2n)\varphi(L_{-m-\mu},Y_m)\\&=(m+\mu-n)\varphi(L_{-m-\mu-n},Y_{m+n})+(2m+2\mu+n)\varphi(L_{-n},Y_{n-\mu}),\\&(m+\mu-n)\varphi(L_{-m-\mu-n},Y_{m+n})\\&=(m+\mu+2n)\varphi(L_{-m-\mu},Y_m)+(2m+2\mu+n)\varphi(L_n,Y_{-n-\mu}).\end{aligned}$$

which together force

$$(2m+2\mu+n)\big(\varphi(L_{-n},Y_{n-\mu})+\varphi(L_n,Y_{-n-\mu})\big)=0, \tag{2.98}$$

and in particular give (by taking $n=2$)

$$(m+\mu+1)\big(\varphi(L_{-2},Y_{2-\mu})+\varphi(L_2,Y_{-2-\mu})\big)=0. \tag{2.99}$$

Noticing $2-\mu>-1-\mu$, $-2-\mu<-1-\mu$, and then combining (2.97) with (2.99), one can safely deduce

$$c_3=c_4. \tag{2.100}$$

Hence (2.97) and (2.100) together give

$$\varphi(L_{-m-\mu},Y_m)=(m+\mu-1)(m+\mu)(m+\mu+1)c_3, \quad \forall\ m\in\mathbb{Z}. \tag{2.101}$$

If $l=-5$, then (2.43) and (2.44) convert to the following form:

$$\begin{aligned}
&(m+\mu-n)\varphi(L_{-m-\mu},Y_m)\\
&=(m+\mu-n)\varphi(L_{-m-\mu-n},Y_{m+n})+(n-m-\mu)\varphi(L_{-n},Y_{n-\mu}), &(2.102)\\
&(m+\mu+2n)\varphi(L_{-m-\mu-n},Y_{m+n})\\
&=(m+\mu+2n)\varphi(L_{-m-\mu},Y_m)-(m+\mu+2n)\varphi(L_n,Y_{-n-\mu}). &(2.103)
\end{aligned}$$

Furthermore, taking $n=-1$ in both (2.102) and (2.103), and using (2.77), one has

$$\begin{aligned}
&(m+\mu+1)\big(\varphi(L_{-m-\mu},Y_m)-\varphi(L_{1-m-\mu},Y_{m-1})\big)=0,\\
&(m+\mu-2)\big(\varphi(L_{-m-\mu},Y_m)-\varphi(L_{1-m-\mu},Y_{m-1})\big)=(m+\mu-1)\varphi(L_{-1},Y_{1-\mu}),
\end{aligned}$$

from which and using (2.77) again, can we deduce the following relation:

$$\varphi(L_{-m-\mu},Y_m)=0, \quad \forall\ m\in\mathbb{Z}. \tag{2.104}$$

Step 2. *The computations of $\varphi(L_{-2\mu-m},M_m)$ and $\varphi(Y_{-2\mu-m},Y_m)$, $\forall\ m\in\mathbb{Z}$.*

Using the similar techniques used in Case 2, one also can obtain the results listed below. If $l=-2$, then

$$\varphi(L_{-m-2\mu},M_m)=\varphi(Y_{-m-2\mu},Y_m)=0, \quad \forall\ m\in\mathbb{Z}. \tag{2.105}$$

If $l=-1$, then

$$\varphi(L_{-m-2\mu},M_m)=\begin{cases}(m+2\mu)\psi(L_1,M_{-1-2\mu}) & \text{if } m\geq-2\mu,\\ (m+2\mu+2)\varphi(L_1,M_{-1-2\mu}) \\ -(m+2\mu+1)\varphi(L_2,M_{-2-2\mu}) & \text{if } m<-2\mu,\end{cases} \tag{2.106}$$

$$\varphi(L_{-m-2\mu},Y_m)=\begin{cases}\big(\frac{m}{\mu}+1\big)\varphi(Y_{-2\mu},Y_0)\\ -m(\frac{m}{\mu}+2)\psi(L_1,M_{-1-2\mu}) & \text{if } m\geq 0,\\ \big(\frac{m}{\mu}+1\big)\varphi(Y_{-2\mu},Y_0)-(\frac{m}{\mu}+2)\big((m+2)\\ \times\varphi(L_1,M_{-1-2\mu})-(m+1)\varphi(L_2,M_{-2-2\mu})\big) & \text{if } m<0.\end{cases} \tag{2.107}$$

If $l=0$, then

$$\begin{aligned}
&\varphi(L_{-m-2\mu},M_m)\\
&=\frac{(m+2\mu)(m+2\mu+1)}{2}\varphi(L_{-1},M_{1-2\mu}),\quad \forall\, m\in\mathbb{Z}; &(2.108)\\
&\varphi(Y_{-4\mu},Y_{2\mu})=\frac{5}{3}\varphi(Y_{-2\mu},Y_0)+\frac{4\mu(4\mu+1)}{3}\varphi(L_{-1},M_{1-2\mu}), &(2.109)\\
&\varphi(Y_{-m-2\mu},Y_m)\\
&=\frac{m(m+1)(m+2\mu)}{m-2\mu}\varphi(L_{-1},M_{1-2\mu})-\frac{3m+2\mu}{m-2\mu}\varphi(Y_{-2\mu},Y_0),\ \text{ if } m\neq 2\mu. &(2.110)
\end{aligned}$$

If $l=1$, then

$$\begin{aligned}
&\varphi(Y_{-\mu-2\mu},Y_\mu)=\frac{1}{4}\mu(\mu^2-1)c_1'-\frac{5}{4}\varphi(Y_0,Y_{-2\mu}), &(2.111)\\
&\varphi(Y_{-m-2\mu},Y_m)=\frac{2m+\mu}{\mu-m}\varphi(Y_{-2\mu},Y_0),\quad \text{if } m\neq\mu; &(2.112)\\
&\varphi(L_{-m-2\mu},M_m)=(m+2\mu-1)(m+2\mu)(m+2\mu+1)c_1',\ \ \forall\, m\in\mathbb{Z}. &(2.113)
\end{aligned}$$

where c_1' is some constant in $\mathbb{C}$. If $l\notin\{-2,\,-1,\,0,\,1\}$, then $\varphi(L_{-m},M_{m-2\mu})=0,\ \forall\, m\in\mathbb{Z}$. If $\frac{2\mu}{l+1}\notin\mathbb{Z}$, then

$$\varphi(Y_{-m-2\mu},Y_m)=-\frac{m(l+3)+2\mu}{m(l+1)-2\mu}\varphi(Y_{-2\mu},Y_0). \tag{2.114}$$

If $\frac{2\mu}{l+1}\in\mathbb{Z}$, $l\notin\{-3,\,-2,\,-1,\,0,\,1\}$, then

$$\varphi(Y_{-m-2\mu},Y_m)=0,\quad \forall\, m\in\mathbb{Z}. \tag{2.115}$$

If $l=-3$, then

$$\varphi(Y_{-m-2\mu},Y_m)=\begin{cases} 0 & \text{if } \ m\geq\mu,\\ -\frac{1}{2}\varphi(Y_0,Y_{-2\mu}) & \text{if } \ m=\mu. \end{cases} \tag{2.116}$$

Step 3. *The computation of* $\varphi(Y_{-3\mu-m},M_m)$, $\forall\, m\in\mathbb{Z}$.

Replacing the triple (v_1,v_2,v_3) by $(L_{-2\mu-m},\,Y_{-\mu},\,M_m)$ in (1.4), one has

$$(m+2\mu)(1+l)\big(\varphi(Y_{-m-3\mu},M_m)+2\varphi(Y_{-\mu},M_{-2\mu})\big)=0,$$

which gives, if $l\neq-1$,

$$\varphi(Y_{-m-3\mu},M_m)=\begin{cases} \varphi(Y_{-\mu},M_{-2\mu}) & \text{if } \ m=-2\mu,\\ -2\varphi(Y_{-\mu},M_{-2\mu}) & \text{if } \ m\neq-2\mu. \end{cases} \tag{2.117}$$

Replacing the triple (v_1, v_2, v_3) by $(L_{-m},\, Y_{-3\mu},\, M_m)$ in (1.4), one has

$$\big(m(1+l)-4\mu\big)\varphi(Y_{-m-3\mu}, M_m) = -2\big(m(1+l)+2\mu\big)\varphi(Y_{-3\mu}, M_0),$$

which can be rewritten as follows, if $l=-1$,

$$\varphi(Y_{-m-3\mu}, M_m) = \varphi(Y_{-3\mu}, M_0) \quad \text{(as } \mu\in\mathbb{Z}^* \text{ in this case).} \tag{2.118}$$

Step 4. *The computation of* $\varphi(M_{-4\mu-m}, M_m)$, $\forall\, m\in\mathbb{Z}$.

Finally, similar as that of Subcase 1.1. we can prove

$$\varphi(M_{-4\mu-m}, M_m) = 0, \quad \forall\, m\in\mathbb{Z}. \tag{2.119}$$

Since for any $x, y, z \in \mathcal{L}_{l,\mu}$, the Jacobian identity must be satisfied, we can obtain all the compatible cocycles. They are just those listed in Lemma 2.3 in this case. This completes the proof of the lemma. □

Then Theorem 1.1 (iv)–(vi) follow from Lemma 2.3. And above all, the main theorem can be easily deduced from lemma 2.1–2.3.

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